The Air Force Research Laboratory (AFRL)

A Pythagorean Excursion:

Ancient Geometry for Advanced Students

Wright-Patterson
Educational Outreach

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Acknowledgement

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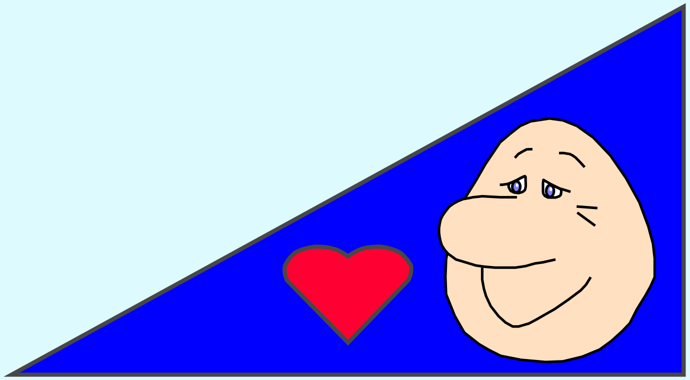
Poem: “Love Triangle”

Consider ol’ Pythagorus,
A Greek of long ago,
And all that he did give to us,
Three sides whose squares now show

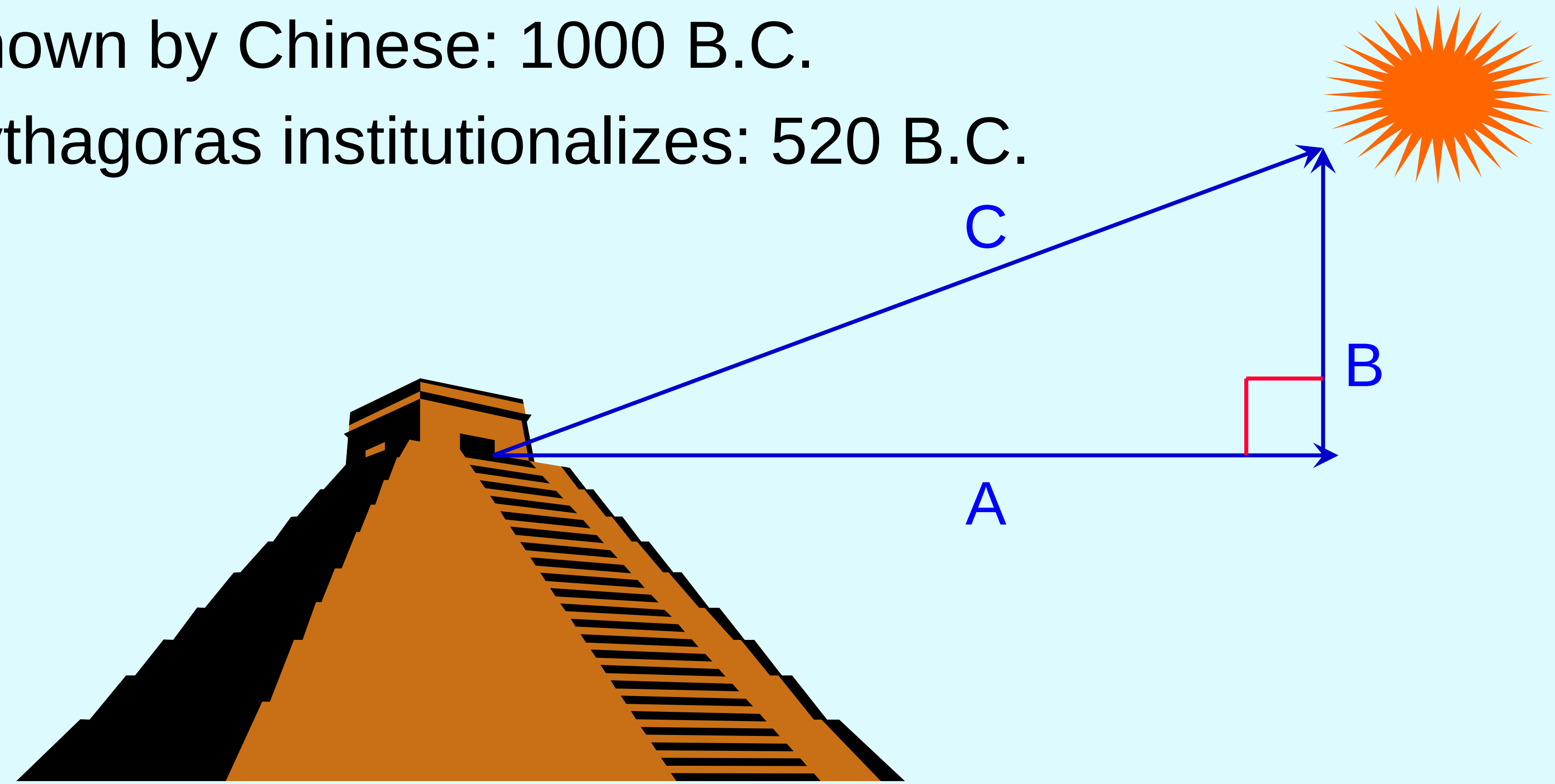
In houses, fields, and highways straight;
In buildings standing tall;
In mighty planes that leave the gate;
And, micro-systems small.

Yes, all because he got it right
When angles equal ninety---
One geek (BC), his plane delight--
One world changed aplenty!

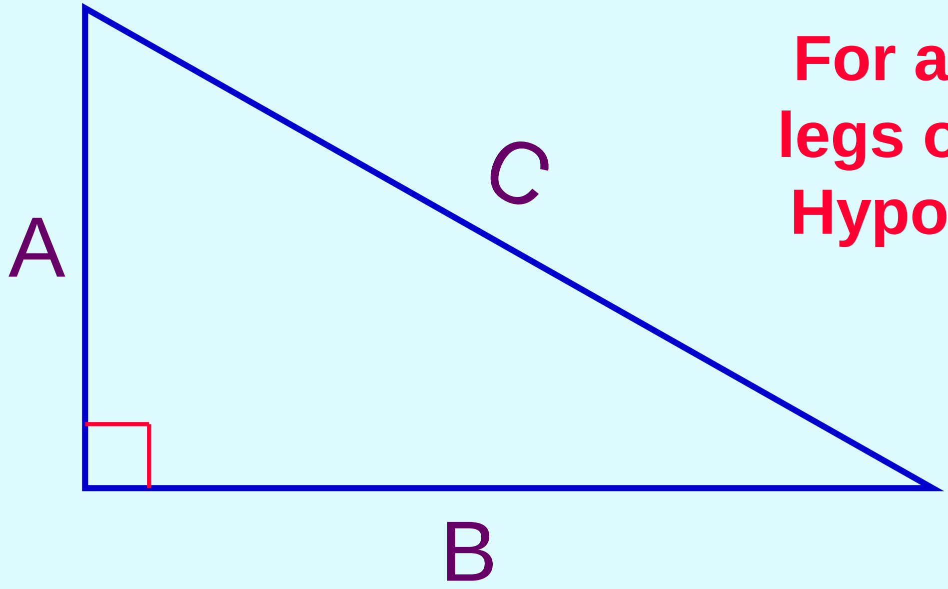


A Very Ancient Discovery: $A^2+B^2=C^2$

- Known by Babylonians: 2000 B.C.
- Known by Chinese: 1000 B.C.
- Pythagoras institutionalizes: 520 B.C.



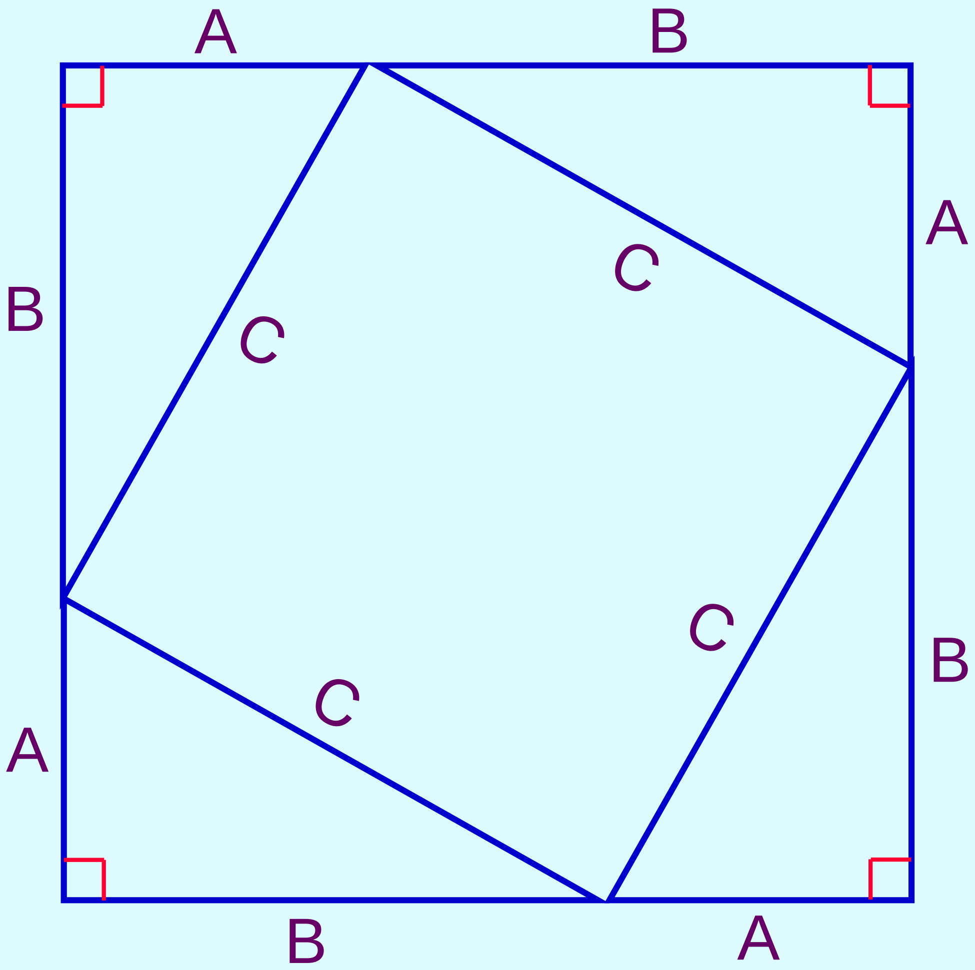
The Pythagorean Theorem



For a right triangle with legs of lengths A, B, and Hypotenuse of length C, $A^2 + B^2 = C^2$.

Pythagoras (569-500 B.C.) was born on the island of Samos in Greece. He did much traveling throughout Egypt learning mathematics. This famous theorem was known in practice by the Babylonians at least 1400 years before Pythagoras!

An Old Proof from China: Circa 1000 B.C.



Proof:

Equate Square Areas

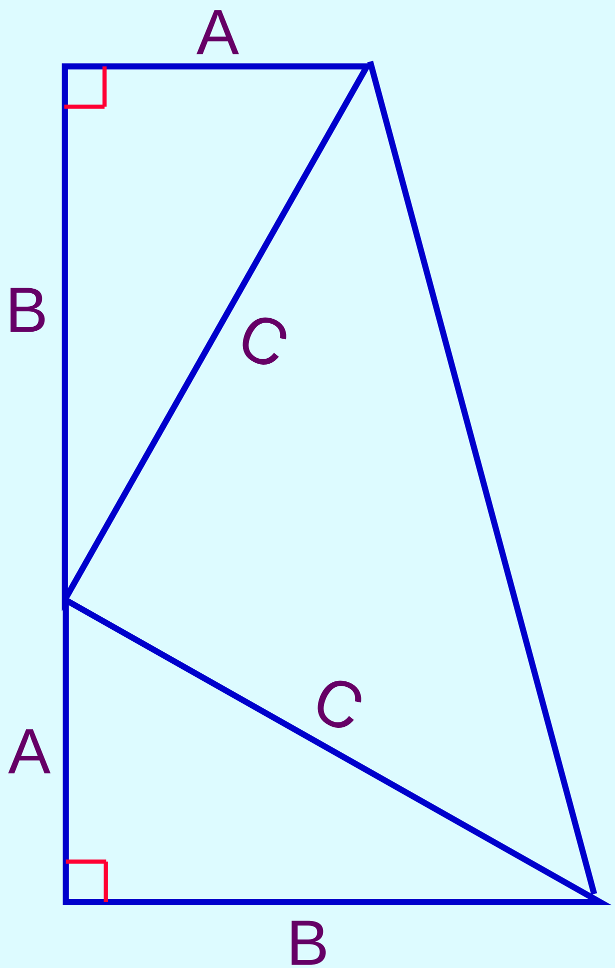
$(A+B)^2 = C^2 + 4(1/2)AB$

$A^2 + 2AB + B^2 = C^2 + 2AB$

$\therefore A^2 + B^2 = C^2$

President Garfield's Proof:

Done while still a Congressman



Proof:

Equate Trapezoidal Areas

$$(1/2)(A+B)^2 = (1/2)C^2 + 2(1/2)AB$$

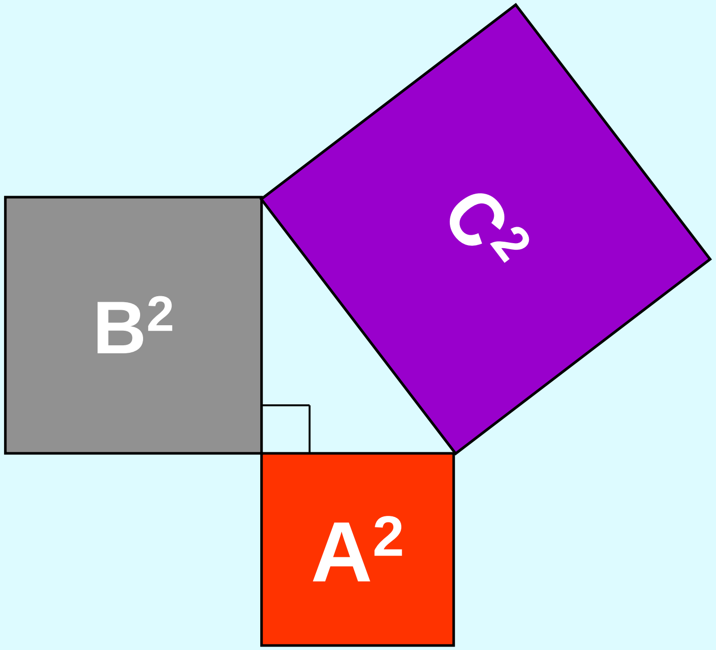
$$(A+B)^2 = C^2 + AB$$

$$A^2 + AB + B^2 = C^2 + AB$$

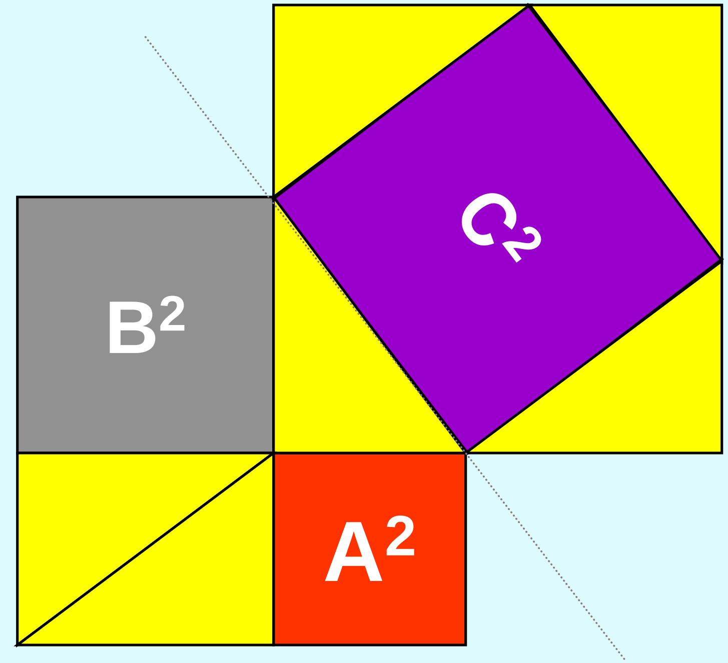
$$:: A^2 + B^2 = C^2$$

Fact: Today there are over 300 known proofs of the Pythagorean theorem!

A Visual Proof of the Pythagorean Theorem

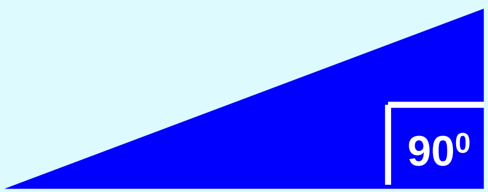


$$C^2=A^2+B^2$$



Q.E.D.

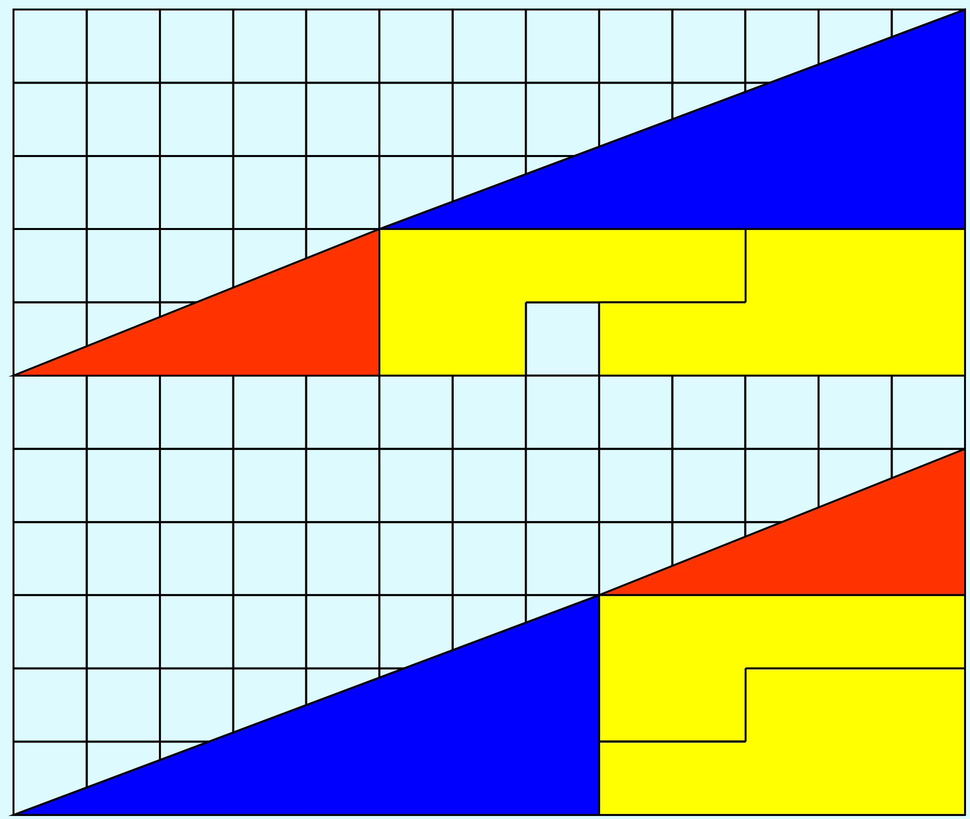
Right Triangles: The Essential Starting Point



Curry's Paradox

Paul Curry was a amateur magician in New York City. In 1953, Paul invented the puzzle shown on the right. I call it "Four Easy Pieces".

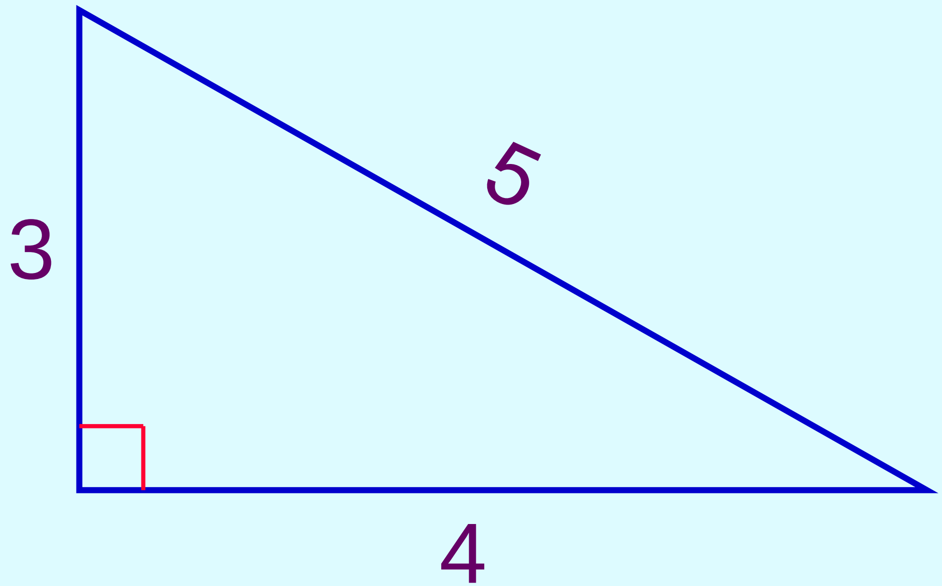
Question: where did the little gap go?



Pythagorean Triangles

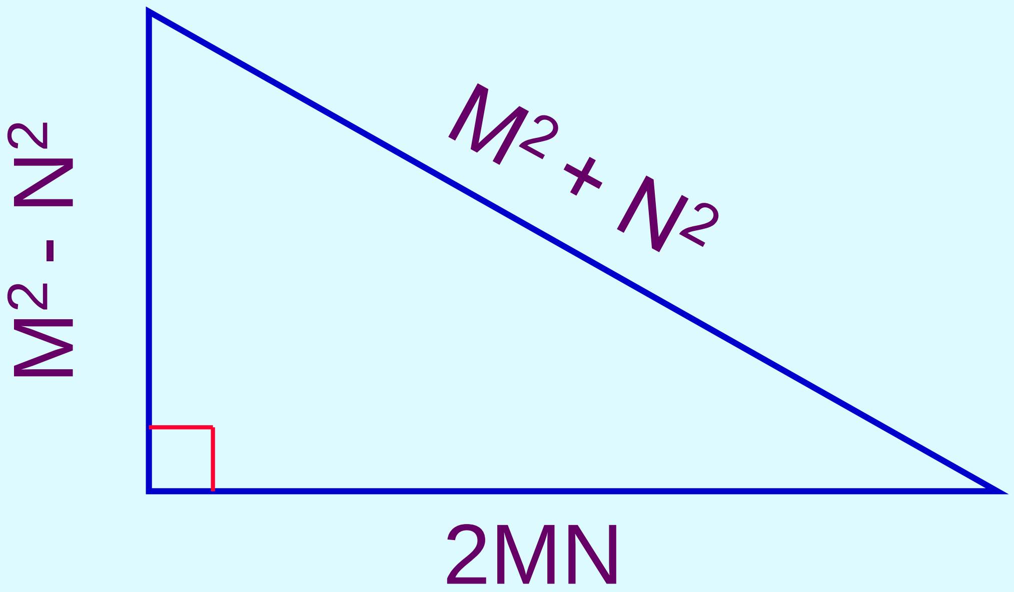
The 3-4-5 triangle

- Only Pythagorean triangle whose sides are in arithmetic progression
- Only triangle of any shape having a perimeter equal to double its area



Pythagorean triangles are right triangles where the lengths of all three sides are expressible as whole numbers.

A Method for Generating Pythagorean Triangles



This method will produce Pythagorean triangles for any pair of whole numbers M and N where M > N. *Question: Can you prove that this method works in general?*

The First Five Pythagorean Triangles--Successive N, M

N	M	A = 2MN	B = $M^2 - N^2$	C= $M^2 + N^2$
1	2	4	3	5
2	3	12	5	13
3	4	24	7	25
4	5	40	9	41
5	6	60	11	61

Notice that one leg and hypotenuse are one unit apart.
Question: Is this always true when M = N + 1?

Ten Pythagorean Triangles: One Leg Equal to 48

A	B	C
48	64	80
48	36	60
48	90	102
48	20	52
48	140	148

Are there any more with A = 48?

A	B	C
48	189	195
48	286	290
48	14	50
48	55	73
48	575	577

Question: How many Pythagorean Triangles can you find where A = 120?

Pythagorean Triangles Having Equal Areas

A	B	C	Area
40	42	58	840
24	70	74	840
15	112	113	840
105	208	233	10920
120	182	218	10920
56	390	392	10920

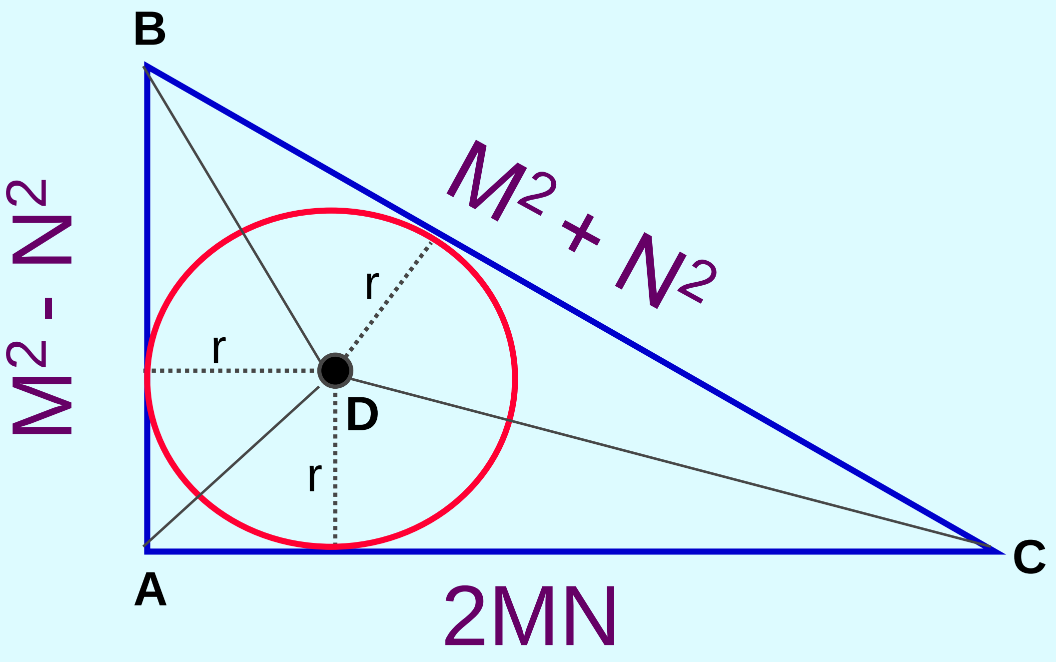
Challenge: Can you find another triplet?

Pythagorean Triangle Rarities: Equal Perimeters

A	B	C	Perimeter (P)
153868	9435	154157	317460
99660	86099	131701	317460
43660	133419	140381	317460
13260	151811	152389	317460

Challenge: 6 other sets of 4 exist where P< 1,000, 000.
Can you find them all?

Pythagorean Triangles: Inscribed Circle Theorem



Let ABC be a Pythagorean triangle generated by the two integers M and N as shown above, where M > N. Then the radius of the corresponding inscribed circle is also an integer.

Proof of the Inscribed Circle Theorem

1) Area ABC = Area ADB + Area BDC + Area CDA

2) $(1/2)\ (M^2-N^2)\ (2MN) = (1/2)r(M^2-N^2) + (1/2)r(M^2+N^2) + (1/2)r(2MN)$

3) $(M^2-N^2)\ (2MN) = r(M^2-N^2) + r(M^2+N^2) + r(2MN)$

4) $r(M^2-N^2) + r(M^2+N^2) + r(2MN) = (M^2-N^2)\ (2MN)$

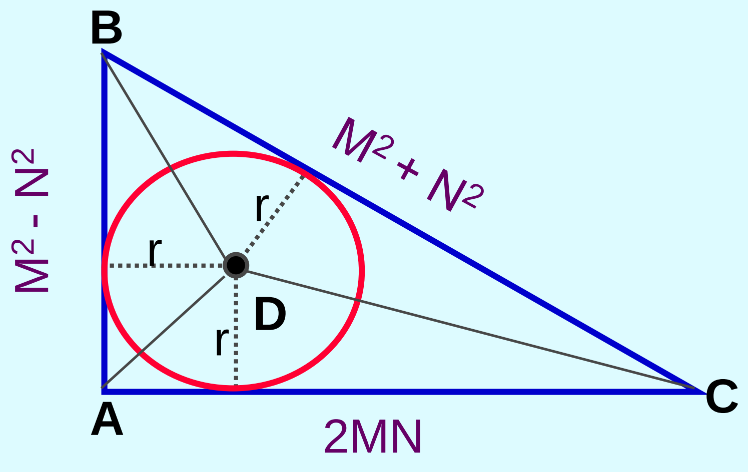
5) $r(2M^2 + 2MN) = (M^2-N^2)\ (2MN)$

6) $r(M^2 + MN) = (M^2-N^2)\ (MN)$

7) $r(M)(M+N) = (M+N)(M-N)(MN)$

Now divide through by M(M+N)

8) $\therefore r = N(M-N)$, an integer



Examples of the Inscribed Circle Theorem

N	M	A	B	C	r
1	2	4	3	5	1
2	3	12	5	13	2
2	4	16	12	20	4
3	5	30	16	34	6

Questions:

1) Can the length of the radius be any positive integer 1, 2, 3, 4…?

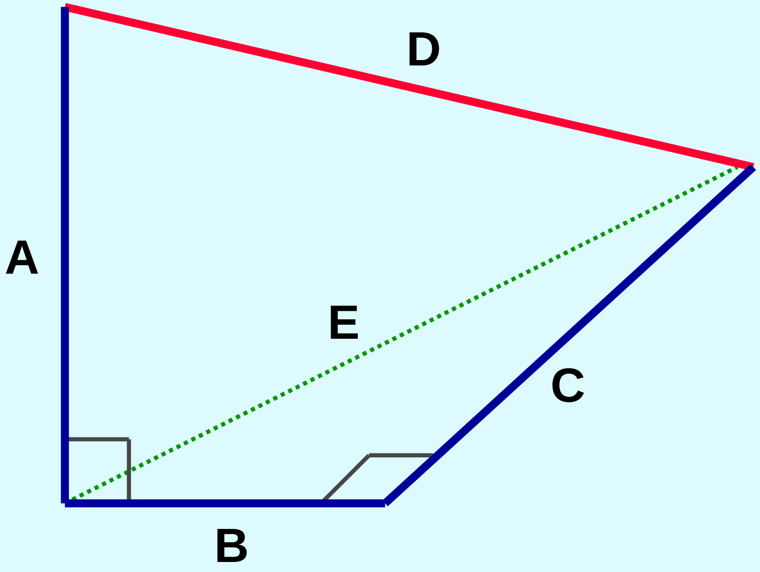
2) Can two Pythagorean triangles each have an inscribed circle having the same radius? More than two?

The Triangular Numbers

					o
				o	oo
			o	oo	ooo
		o	oo	ooo	oooo
	o	oo	ooo	oooo	ooooo
o	oo	ooo	oooo	ooooo	oooooo
1	**3**	**6**	**10**	**15**	**21**

Above is the process for making a partial sequence of triangular numbers. As you can see, triangular numbers are well-named! The full sequence is the infinite sequence 1, 3, 6, 10, 15, 21, 28, 36, 45...

The Pythagorean Theorem in Three Dimensions



For a right-triangular (orthogonal) three-dimensional system:
$D^2 = A^2 + B^2 + C^2$

Proof:

1) $D^2 = A^2 + E^2$

2) But $E^2 = B^2 + C^2$

3) :: $D^2 = A^2 + B^2 + C^2$

This result extends to n dimensions by the simple process of mathematical induction. The multi-dimensional Pythagorean Theorem forms the basis of Analysis of Variance (ANOVA), a powerful modern statistical analysis technique formulated by Sir Ronald Fisher in the 1920s.

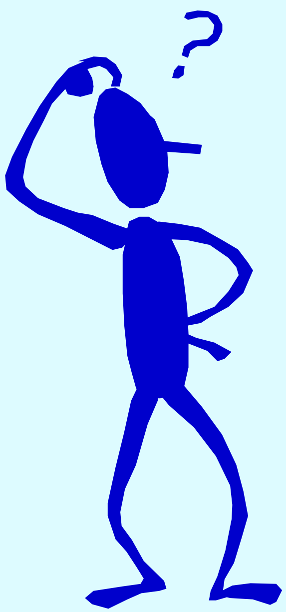
Fermat's Last Theorem:
You Don't Mess With the Exponent!

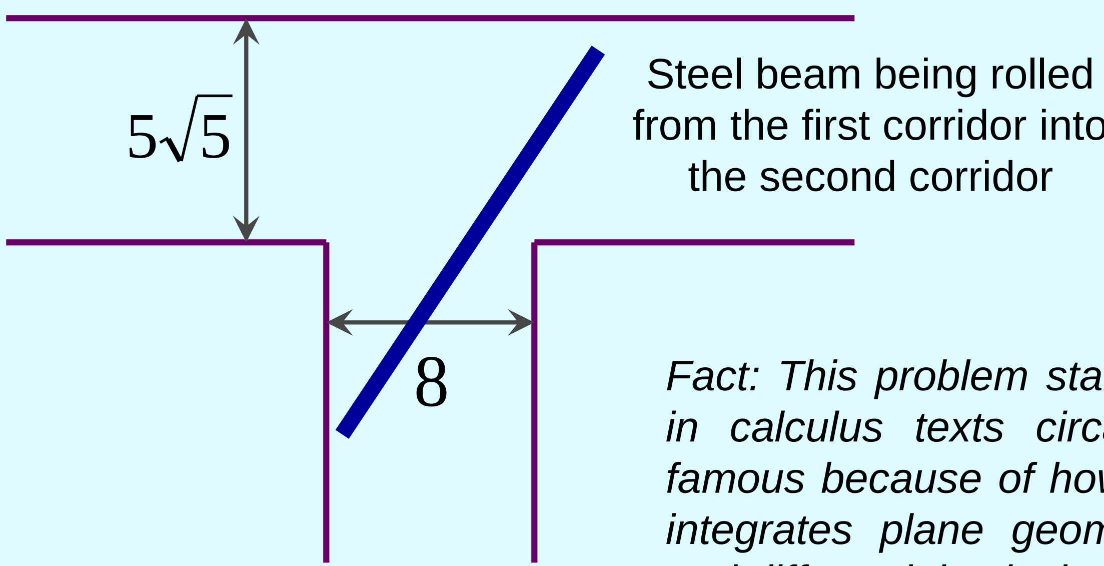
Hypothesis: Let n be a whole number 2, 3, 4, 5…
Then, $X^n + Y^n = Z^n$ has no solution in integers or rational fractions except for the case n = 2, the Pythagorean case.
This "Anti-Pythagorean" extension was first proposed by Fermat circa 1660 and finally proved to be true by Andrew Wiles of Princeton in 1993.

Calculus Meets Pythagoras:

The Infamous Girder Problem!

Two workmen at a construction site are rolling steel beams down a corridor *8* feet wide that opens into a second corridor $5\sqrt{5}$ feet wide. What is the length of the longest beam that can be rolled into the second corridor? Assume that the second corridor is perpendicular to the first corridor and that the beam is of negligible thickness.





Fact: This problem started to appear in calculus texts circa 1900. It is famous because of how it thoroughly integrates plane geometry, algebra, and differential calculus.

Answer: 27 feet

Thank You!