Add It Up!

With

Magic Squares

By Wizard John

2	7	6
9	5	1
4	3	8

Produced and Published by the:

WPAFB Educational Outreach Office
Building 196
Wright-Patterson AFB, Ohio 45433-7542

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O Icarus...

I ride high...
With a whoosh to my back
And no wind to my face,
Folded hands
In quiet rest—
Watching...O Icarus...
The clouds glide by,
Their fields far below
Of gold-illumed snow,
Pale yellow, floating moon
To my right—evening sky.

And Wright...O Icarus...
Made it so—
Silvered chariot streaking
On tongues of fire leaping—
And I will soon be sleeping
Above your dreams...J. Sparks 2001

100th Anniversary of
Powered Flight:
1903—2003

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Table of Contents

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1) The 3x3 Magic Square

1.1 Why This Square is Magic

Draw a large blank square having three rows and three columns. This splits the big square into nine little squares. Then enter the first nine counting numbers as shown:

2	7	6
9	5	1
4	3	8

The result is called a **3x3 magic square** (read "three by three"). Why is this square magic? Add the three numbers in any one row, or any one column, or along any one of the two diagonals. If you do this, you will get the exactly the same number, which is **15**. The **number 15** is called the **magic sum**. The three side-by-side magic squares below show how the three numbers add up to **15** for the first row, the third column, and the downward diagonal going from left to right.

Add Up a Row

2	7	6
9	5	1
4	3	8

Add Up a Column

2	7	6
9	5	1
4	3	8

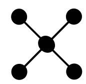
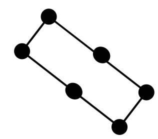
Add Up a Diagonal

2	7	6
9	5	1
4	3	8

Activity: Find the total number of ways that this square adds up to 15.

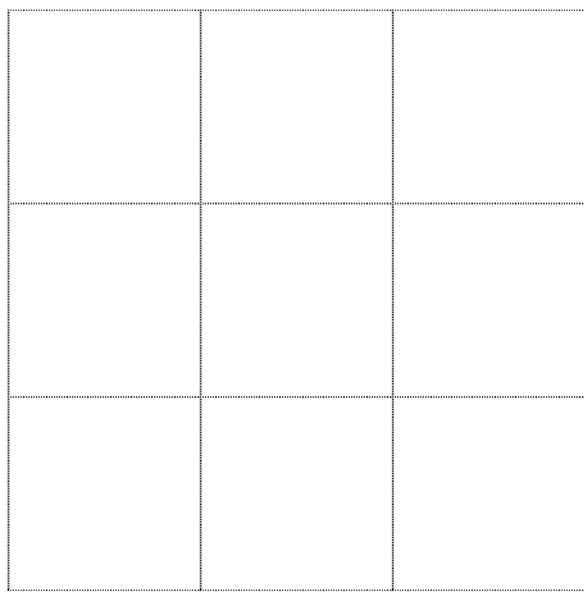
Magic squares have been around for a long time. Most historians agree that they originated in China many centuries before the birth of Christ. Chinese tradition holds that the Emperor Yu saw a mystical three by three pattern on the back of a tortoise as it sunned itself on the bank of the Yellow river. The time that this happened was about 2200 B.C. The pattern that Emperor Yu saw became known as the **lo-shu**, which had a pretty arrangement of dots in each of the nine little squares.

The number of dots in each little lo-shu square is equal to the actual number in the same little square today. Two of the lo-shu patterns are

which represent the numbers 5 and 6. These little groupings of dots are what legend claims Emperor Yu saw on the back of the tortoise.

Activity: Create a "lo-shu style" 3x3 magic square in the blank below. Represent the numbers 1 through 9 by cool patterns of dots, stars, happy faces, etc.



My "lo-shu" Magic Square

1.2 Challenges for Those in Grade School

1. Add up all the numbers in the 3x3 magic square. What is the total?
2. Divide the total obtained in Question 1 by 3. What number do you get as an answer? Do you recognize this number? Why do you think it appeared?
3. Using nine consecutive counting numbers, create a 3x3 magic square where the three rows, the three columns, and the two diagonals all add up to the magic sum **18**.
4. Using nine consecutive counting numbers, create a 3x3 magic square where the three rows, the three columns, and the two diagonals all add up to the magic sum **48**.

1.3 Additional Challenges for Those in Middle School

1. Can you make a 3x3 magic square having a magic sum equal to **100** using nine consecutive counting numbers? Why or why not?

Pure or Normal Magic Squares are magic squares where the numbers in the little squares are consecutive whole numbers starting with the number 1. The magic square that you are to create in Challenge 2 is not a pure magic square.

2. Create a 3x3 magic square using the nine prime numbers 5, 17, 29, 47, 59, 71, 89, 101, and 113. *Hint: First determine the magic sum by adding up all the numbers and dividing by _______?*

3. Show that you can not make a 3x3 magic square using the first nine prime numbers: 2,3,5,7,11,13,17,19,23.

4. Explain why the book starts with a 3x3 magic square. *Hint: Show that it is impossible to make a 2x2 pure magic square or any sort of 2x2 magic square.*

☺	☹
☹	☺

Am I Possible?

5. Show that the sum of the squares in the first row is equal to the sum of the squares in the third row (i.e. $2^2+7^2+6^2=4^2+3^2+8^2$). What is the common sum? Likewise, show that the sum of the squares in the first column is equal to the sum of the squares in the last column.

1.4 Additional Challenges for Those in High School

1. Prove that the number in the center of the 3x3 magic square must be 5.
2. Prove that the magic sum for a NxN pure magic square is given by the expression $\frac{N(1+N^2)}{2}$. *Hint: First show that the sum of the first n consecutive integers is given by* $\frac{n(n+1)}{2}$.

2) The 4x4 Magic Square

2.1 Bigger Means More Magic

After several centuries, the 3x3 magic square made its way out of China and entered the Indian subcontinent. From India, it traveled on to Arabia and eventually into medieval Europe. During its journey, ancient mathematicians discovered that the 3x3 magic square could be increased in size to a **4x4 Magic Square**. The mathematicians also discovered—unlike the 3x3 pure magic square—that there was *more than one way* of constructing a 4x4 *pure* magic square using the first 16 counting numbers 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16. Below are two *out of 880* possible such patterns.

2	7	11	14
16	9	5	4
13	12	8	1
3	6	10	15

16	3	2	13
5	10	11	8
9	6	7	12
4	*15*	*14*	1

The magic square on the right is a very famous 4x4 magic square. It first appeared in an engraving entitled “Melancholia”. The engraving was done by the German Albrecht Durer in the year 1514 and is shown to the right. Can you see the date in the square? 4x4 magic squares are just like 3x3 magic squares in that the four numbers in each row, in each column, and in each diagonal all add up to the same magic sum.

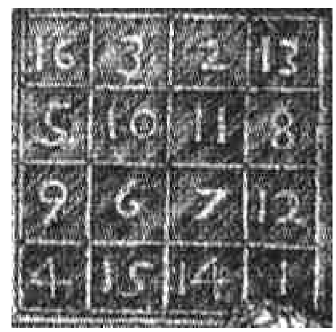
Here are three starter activities:

1. *Find the magic sum for a 4x4 magic square*
2. *Find the total number of ways rows, columns, and diagonals can add up to the magic sum*
3. *Sum all the numbers in the 4x4 magic square. How many times the magic sum is this?*

2.2 Challenges for Those in Grade School

1. Since there are 280 different ways of making a 4x4 pure magic square, can you create a 4x4 magic square not shown in this booklet?
2. Make a 4x4 magic square (not a pure magic square) whose magic sum is 38.
3. Make a 4x4 magic square whose magic sum is 78.
4. Make a 4x4 magic square whose magic sum is 102.

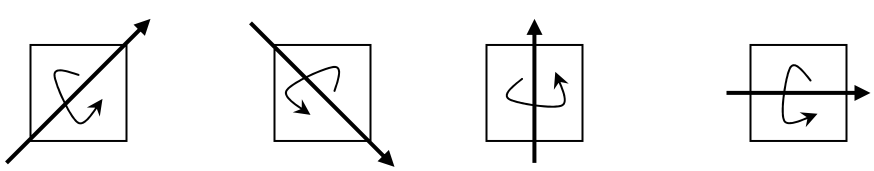
2.3 Additional Challenges for Those in Middle School



1. In the Albrecht Durer magic square (enlarged to the right as it appears in the engraving Melancholia), show that the sum of the squares in the first row is equal to the sum of the squares in the forth row (i.e. $16^2+3^2+2^2+13^2=4^2+15^2+14^2+1^2$). Show the sum of the squares in the second row is equal to the sum of the squares in the third row.
2. Show that a similar sum-of-the-squares equality holds for the columns of the Durer square.
3. Finally, does a similar sum-of-squares equality exist for the numbers on the two diagonals?
4. Does this same overall sum-of-squares property hold for the magic square pictured to the left of the Durer square on the previous page? For rows? For columns? For diagonals?

2.4 Additional Challenges for Those in High School

1. Given that you have a 4x4 magic square, how many new magic squares can be generated from the given square by doing one or more of the following: interchanging rows, interchanging columns, and reflecting the square in one of four different ways as shown below?



3) The Perfect 4x4 Magic Square

The 4x4 magic square below "adds up" its rows, columns, and diagonals as it should. But, it adds up in a many more ways as we shall now illustrate—making the pure perfect!

1	15	6	12
8	10	3	13
11	5	16	2
14	4	9	7

3.1 Add Up More Squares and Diagonals

First, we will review the three necessary ways by simply using the definition of what it means to be a magic square.

Rows

1	15	6	12
8	10	3	13
11	5	16	2
14	4	9	7

Columns

1	15	6	12
8	10	3	13
11	5	16	2
14	4	9	7

Diagonals

1	15	6	12
8	10	3	13
11	5	16	2
14	4	9	7

Below are three new ways. All four corners associated with any 2x2, 3x3, or 4x4 square embedded within our big square also adds up to the magic sum.

2x2 Squares

1	15	6	12
8	10	3	13
11	5	16	2
14	4	9	7

3x3 Squares

1	15	6	12
8	10	3	13
11	5	16	2
14	4	9	7

One 4x4 Square

1	15	6	12
8	10	3	13
11	5	16	2
14	4	9	7

Here are some more new ways. "Broken Diagonals" also add up to the same magic sum.

Broken Diagonal

1	15	6	12
8	10	3	13
11	5	16	2
14	4	9	7

Broken Diagonal

1	15	6	12
8	10	3	13
11	5	16	2
14	4	9	7

Broken Diagonal

1	15	6	12
8	10	3	13
11	5	16	2
14	4	9	7

Broken Diagonal

1	15	6	12
8	10	3	13
11	5	16	2
14	4	9	7

Broken Diagonal

1	15	6	12
8	10	3	13
11	5	16	2
14	4	9	7

Broken Diagonal

1	15	6	12
8	10	3	13
11	5	16	2
14	4	9	7

Notice that each new pattern still adds up four numbers to achieve the same magic sum.

3.2 Challenges for Those in Grade School

1. Find the total number of ways that all the patterns shown (including the three necessary patterns) add up to the magic sum.
2. Can you find any other neat patterns where four numbers in the pattern add up to the magic sum?

3.3 Additional Challenges for Those in Middle School

1. How many of the sum-of-squares equalities still hold with respect to rows, columns, and diagonals associated with our perfect 4x4 magic square?
2. How many additional sum-of-square equalities can you discover within our new "add-it-up" patterns?

3.4 Additional Challenges for Those in High School

1. By adding up all the various possibilities, show that the Albrecht Durer square (below) is a perfect 4x4 magic square.

16	3	2	13
5	10	11	8
9	6	7	12
4	*15*	*14*	1

2. Show that the sum-of-squares for the eight numbers on the two diagonals equals the sum-of-squares for the eight numbers *not on* the two diagonals.
3. Show that the sum-of-cubes for the eight numbers on the two diagonals is equals the sum-of-cubes for the eight numbers *not on* the two diagonals (hint: the sum of cubes for the first row is given by $16^3 + 3^3 + 2^3 + 13^3$).
4. Are there any other patterns of any sort that you can discover within Durer's awesome square?

A 4x4 **heterosquare** is just the opposite of a magic square in that none of the rows, columns, or diagonals sum to the same number. Below is a 3x3 heterosquare that actually took a long time to be discovered. Charles Trigg was the recreational mathematician who finally discovered it.

9	8	7
2	1	6
3	4	5

5. 4x4 heterosquares were in existence long before the 3x3 heterosquare was discovered by Charles Trigg. Your challenge is to construct one of these 4x4 heterosquares.

4) The Perfect 5x5 Magic Square

Rows, columns, diagonals, and broken diagonals all add up to the same magic sum in the perfect 5x5 magic square—*shown with shaded master star*.

1	15	24	8	17
23	7	16	5	14
20	4	13	22	6
12	21	10	19	3
9	18	2	11	25

Activity: Find the magic sum for this magic square. Find the total number of ways that rows, columns, diagonals, and broken diagonals add up to the magic sum.

4.1 Add Up Stars and Rhomboids

There are several new patterns associated with the 5x5 magic square where numbers total to the magic sum. These patterns are center stars, corner stars and the one master star. Representative center and corner stars are shown below. The master star is shaded in the 5x5 magic square shown above.

A Center Star

1	15	24	8	17
23	7	16	5	14
20	4	13	22	6
12	21	10	19	3
9	18	2	11	25

A Corner Star

1	15	24	8	17
23	7	16	5	14
20	4	13	22	6
12	21	10	19	3
9	18	2	11	25

Activity: Find the total number of additional ways the perfect 5x5 magic square will add up to the magic sum by counting all corner stars, center stars, and the one master star. Be sure to check the magic sum for each star by "adding it up"!

In addition to stars, there are pretty rhomboids that also add up to the magic sum. Four representatives of these rhomboids are shown below. All rhomboids must include the number in the center (for a total of five numbers) in order to add up to the magic sum. The master star is sometimes called a 4x4 rhomboid. Also, the slanted 3x3 rhomboid is sometimes called a **centered master star**.

A Slanted 2x2 Rhomboid

1	15	24	8	17
23	7	16	5	14
20	4	13	22	6
12	21	10	19	3
9	18	2	11	25

One Slanted 3x3 Rhomboid

1	15	24	8	17
23	7	16	5	14
20	4	13	22	6
12	21	10	19	3
9	18	2	11	25

Another Slanted 2x2 Rhomboid

1	15	24	8	17
23	7	16	5	14
20	4	13	22	6
12	21	10	19	3
9	18	2	11	25

A Parallel 3x3 Rhomboid

1	15	24	8	17
23	7	16	5	14
20	4	13	22	6
12	21	10	19	3
9	18	2	11	25

4.2 Challenges for Those in Grade School

1. Add up all the rhomboid patterns that total to the magic sum.
2. Add up all the patterns presented in this section: rows, columns, diagonals, broken diagonals, corner stars, center stars, the one master star, and various kinds of rhomboids.
3. Make a 5x5 magic square whose magic sum is 100.
4. Can you find any other neat patterns where *five numbers* in the pattern add up to the magic sum?

4.3 Additional Challenges for Those in Middle School

1. How many of the sum-of-squares equalities still hold with respect to rows, columns, and diagonals associated with our perfect 5x5 magic square?
2. How many additional sum-of-square equalities can you discover within our new “add-it-up” patterns?
3. Break the numbers 1 through 25 into five groups. Let the first group consist of the numbers 1, 2, 3, 4, 5. Let the second group consist of the numbers 6, 7, 8, 9, 10. In like fashion, form the third, forth, and fifth group. Show that every row, column, and main diagonal in the perfect 5x5 magic square contains exactly one number from each group.

4.4 Additional Challenges for Those in High School

1. Construct a 5x5 heterosquare.
2. There are 3600 different 5x5 magic squares. Not all of them are as perfect as the one shown in this booklet. Create a pure 5x5 magic square that satisfies the basic requirements—rows, columns, and the two main diagonals add up—but fails to add up for at least one center star, corner star, rhomboid, etc.

5) A Nested 5x5 Magic Square

This 5x5 magic square has a shaded 3x3 square neatly tucked inside. The 5x5 magic square is a pure magic square whose rows, columns, and two main diagonals all “add up”. The 3x3 square is also a magic square, but not a pure magic square since it uses numbers other than 1, 2, 3, 4, 5, 6, 7, 8, and 9. However, the numbers used in the 3x3 magic square are consecutive, starting with 9 and ending with 16—a wonder to behold!

1	18	21	22	3
2	10	17	12	24
18	15	13	11	8
21	14	9	16	5
23	7	6	4	25

Activity: For younger students, find the magic sum for the 3x3 magic square. For older students, how many “perfect properties” have been lost in this 5x5 magic square?

6) Ben Franklin's 8x8 Magic Square

Benjamin Franklin—American scientist, inventor, statesman, philosopher, economists, musician, and printer—invented this 8x8 magic square in his spare time. It is a pure magic square in that it utilizes the consecutive counting numbers from 1 to 64. He also invented a 16x16 which is much too large to put in this little booklet.

14	3	62	51	46	35	30	19
52	61	4	13	20	29	36	45
11	6	59	54	43	38	27	22
53	60	5	12	21	28	37	44
55	58	7	10	23	26	39	42
9	8	57	56	41	40	25	24
50	63	2	15	18	31	34	47
16	1	64	49	48	33	32	17

Shown are two neat eight-number patterns embedded in Franklin's 8x8 magic square. From these two patterns one can immediately spot two additional patterns (*hint: the two additional patterns are two-tone patterns*).

Activities

For younger students, find the magic sum for Franklin's 8x8 magic square. Verify that all the rows, all the columns, and the two main diagonals indeed add up to the magic sum. How many ways is this?

For older students, explore the 8x8 magic square! Have a class contest to see who can discover the most eight-number patterns embedded in Franklin's 8x8 magic square. How many of the sum-of-squares properties are present in Franklin's 8x8 magic square? If we partition the numbers 1 through 64 into eight consecutive groups—where the first group consists of the numbers 1 though 8, the second group 9 through 16, etc.—is there one and only one number present from each group in every row, column, and main diagonal?

7) A Nested 9x9 Magic Square

This absolutely awesome pure 9x9 magic square, invented by the French mathematician Bernard de Bessy in the 1600s, contains the following magic squares: a 3x3 magic square, a 5x5 magic square, and a 7x7 magic square. Admittedly, the smaller squares are not pure; but all of the squares will still add up to hours of fun and entertainment in your classroom!

16	81	79	78	77	13	12	11	2
76	28	65	62	61	26	27	18	6
75	23	36	53	51	35	30	59	7
74	24	50	40	45	38	32	58	8
9	25	33	39	41	43	49	57	73
10	60	34	44	37	42	48	22	72
14	63	52	29	31	47	46	19	68
15	64	17	20	21	56	55	54	67
80	1	3	4	5	69	70	71	66

Activities

For younger students, find the magic sum for all four squares. Find the total number of ways—rows, columns, and main diagonals—that the 3x3 adds up to its magic sum, the 5x5 adds up to its magic sum, the 7x7 adds up to its magic sum, and the 9x9 adds up to its magic sum. Then, total the totals to get one big

Add It Up!

For older students, explore the 9x9 magic square asking the same questions that we have already asked. Have fun and add it up! One new thing: notice that the number in the middle of de Bessy's 9x9 magic square is 41, halfway between 1 and 81. Likewise, the number in the middle of a pure 5x5 magic square is 13, halfway between 1 and 25. Our little 3x3 magic square exhibits the same property. So what number do you think is in the middle of a pure 7x7 magic square, a pure 11x11 magic square, a pure 13x13 magic square?

8) Two Magic Squares for Reference

A 6x6 Magic Square

1	32	33	34	35	6
12	29	9	10	26	25
13	14	22	21	23	18
24	20	16	15	17	19
30	11	28	27	8	7
31	5	33	4	2	36

A 7x7 Magic Square

22	21	13	5	46	38	30
31	23	15	14	6	47	39
40	32	24	16	8	7	48
49	31	33	25	17	9	1
2	43	42	34	26	18	10
11	3	44	36	35	27	19
20	12	4	45	37	29	28

9) One Great Magic-Square Resource Book*...brand new!*

<u>The Zen of Magic Squares, Circles, and Cubes</u>; Clifford A. Pickover, Princeton University Press, 2002